Symbolic Powers of Ideals: Problems and Progress

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DORDT COLLEGE

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Introduction

Outline:

- Ideals and their powers
- Questions of study
- Recent results

Our situation

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A non-example is $J = (X^2 - Y, Z^2)$.

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Answer: We have $I^r \subseteq I^t$ if and only if $r \geq t$. *ideal gets (strictly) smaller*

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Given an ideal $I \subseteq R$, we define the m-th symbolic power of I to be

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Before elaborating, we ask: what can symbolic powers look like?

Example – Symbolic Powers of Edge Ideals

Introduced by R. Villareal in the 1990s

Let $V = \{x_1, x_2, \ldots, x_n\}$ be a set of variables and consider the (simple) graph $G = (V, E)$, where E contains 2-element sets comprised of pairs of the variables (so, e.g., $\{x_1, x_2\} \in E$ but $\{x_1, x_2, x_3\}, \{x_1^2\} \notin E$).

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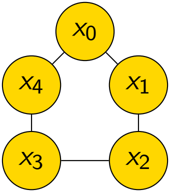
Fact: For an edge ideal I, $I^{(m)} = \bigcap_i P_i^m$, where the P_i correspond to minimal vertex covers of G.

$I = I(C_5)$



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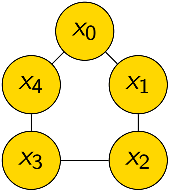

Here, the ring is $R = k[x_0, x_1, x_2, x_3, x_4]$, and the ideals corresponding to minimal vertex covers are $P_1 = (x_0, x_1, x_3)$, $P_2 = (x_0, x_2, x_3)$, $P_3 = (x_0, x_2, x_4)$, $P_4 = (x_1, x_2, x_4)$, $P_5 = (x_1, x_3, x_4)$. Then

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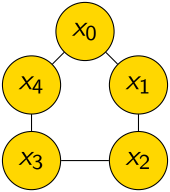

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$$\begin{aligned} I^{(2)} &= P_1^2 \cap P_2^2 \cap P_3^2 \cap P_4^2 \cap P_5^2 \\ &= (x_0^2x_1^2, x_0x_1^2x_2, x_1^2x_2^2, x_0x_1x_2x_3, x_1x_2^2x_3, x_2^2x_3^2, x_0^2x_1x_4, x_0x_1x_2x_4, \\ &\qquad x_0x_1x_3x_4, x_0x_2x_3x_4, x_1x_2x_3x_4, x_2x_3^2x_4, x_0^2x_4^2, x_0x_3x_4^2, x_3^2x_4^2) \\ &= I^2. \end{aligned}$$

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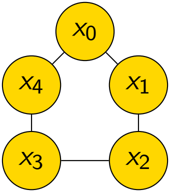

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But $I^{(t)} \neq I^t$ for all $t > 2$.

Equality of the powers

Theorem (Simis-Vasconcelos-Villareal (1994))

Given an edge ideal $I = I(G) \subseteq k[x_1, x_2, \ldots, x_n]$ as above, the following are equivalent.

(i) $I^{(m)} = I^m$ *for all* $m \geq 1$.

(ii) *The graph G is bipartite.*

Example – Points in $\mathbb{P}^N$

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Then $I = \cap_j I(p_j)$ is the ideal generated by all forms vanishing at each p_j.

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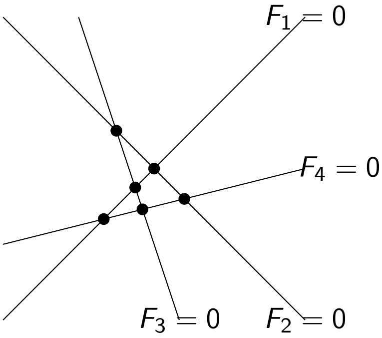
Theorem (Nagata, Zariski)

Let $Z = \{p_1, \ldots, p_r\} \subseteq \mathbb{P}^N_C$. Given $I = I(Z) = \cap_j I(p_j)$,

$$I^{(m)} = \bigcap_{j=1}^{r} I(p_j)^m$$

is the ideal of all forms vanishing to order at least m at each of the p_j's.

An example of points in $\mathbb{P}^2$



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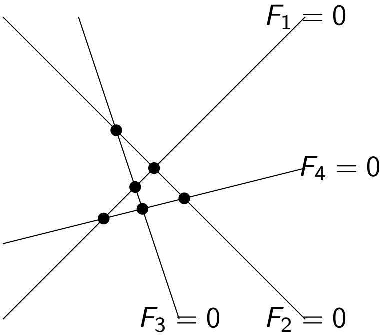

Given $I = I(Z)$, $F_1F_2F_3F_4 \in I^{(2)}$ but no product of 3 of the F_i's are in $I^{(2)}$.

Points in $\mathbb{P}^2$

Let $Z \subseteq \mathbb{P}^2$ be a finite set of points.

Question

If $I = I(Z)$ is generated by 3 or more forms, is it ever true that $I^{(2)} = I^2$?

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The answer seems to be no.

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- If $I = I(G)$ is the edge ideal of a graph G, $I^{(m)} = \cap_i P_i^m$, where the P_i's are generated by the variables corresponding to minimal vertex covers of G.

Symbolic power summary

Our examples:

- If $I = I(G)$ is the edge ideal of a graph G, $I^{(m)} = \cap_i P_i^m$, where the P_i's are generated by the variables corresponding to minimal vertex covers of G.
- If $I = I(Z)$ is the ideal of a finite set of points in $\mathbb{P}^N$, $I^{(m)}$ is the ideal generated by all forms vanishing to order at least m at each point of Z.

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Problem (Containment Problem)

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Problem

Compute invariants related to the containment $I^{(m)} \subseteq I^r$.

The Containment Problem: a general solution

Theorem (Ein-Lazarsfeld-Smith, Hochster-Huneke)

Let k be an algebraically closed field of arbitrary characteristic and suppose $I \subseteq k[x_0, x_1, \ldots, x_N]$ is a nontrivial homogeneous ideal. Then $I^{(m)} \subseteq I^r$ whenever $m \geq Nr$.

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Question

Given additional information about I, can m be made smaller?

Yes!

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If $I = I(G)$ with G bipartite on N vertices, $I^{(m)} \subseteq I^r$ if and only if $m \geq r$.

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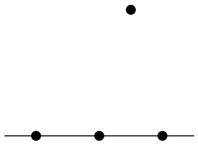
Theorem (–, 2013)

If $I = I(Z)$ is the set of $n+1$ almost collinear points in $\mathbb{P}^2$ (see figure for $n+1=4$), then $I^{(m)} \subseteq I^r$ if and only if

$$m > \frac{n^2 r - n}{n^2 - n + 1}.$$



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Problem (Asymptotic Containment Problem)

Compute $\rho(I)$.

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- There are not many known values of $\rho(I)$ (though for $n+1$ almost collinear points, $\rho(I) = \frac{n^2}{n^2-n+1}$).
- Easier problem: compute invariants which contribute to (better) bounds for $\rho(I)$.

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Of importance in this line of questioning is the *initial degree*.

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Given I, the edge ideal of C_{2n+1},

$$\alpha(I^{(m)}) = 2m - \lfloor \frac{m}{n+1} \rfloor$$

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Definition

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Theorem (Bocci-Harbourne, 2007)

Given a nonzero homogeneous $I \subsetneq k[x_0, x_1, \ldots, x_N]$,

$$\frac{\alpha(I^m)}{\alpha(I^{(m)})} \leq \frac{\alpha(I)}{\hat{\alpha}(I)} \leq \rho(I).$$

Results for edge ideals

Theorem (Bocci, J–, van Tuyl)

Let $I \subseteq k[x_1, x_2, \ldots, x_{2n+1}]$ *be the edge ideal of the cycle* C_{2n+1}. *Then:*

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Let G_1 *and* G_2 *be graphs with disjoint vertex sets* $V_1 = \{x_1, \ldots, x_n\}$ *and* $V_2 = \{y_1, \ldots, y_n\}$, *respectively. Then*

$$I(G_1 \cup G_2)^{(m)} = \sum_{i=0}^{m} I(G_1)^{(i)} I(G_2)^{(m-i)}.$$

Future work

Problem

Compute $\rho(I)$ and $\hat{\alpha}(I)$ for additional ideals.

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What is $\rho(I(G_1 \cup G_2))$?

Thanks

Thank you!