On the fattening of lines in $\mathbb{P}^3$

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Oberwolfach Mini-Workshop

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Of recent interest is the initial sequence $(\alpha(I^{(m)}))_{m\geq 1}$. We will write $\alpha(mZ) := \alpha(I(Z)^{(m)})$.

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Today we examine the situation for lines in $\mathbb{P}^3$.

Stars to Pseudostars

Definition

Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ *be a collection of* $s \geq 1$ *distinct hyperplanes in* $\mathbb{P}^N$*. We assume that the intersection of any* j *of the hyperplanes is either empty or has codimension* j*. For any* $1 \leq c \leq \min(s, N)$ *define:*

$$V_c(\mathcal{H}, \mathbb{P}^N) = \bigcup_{1 \leq i_1 < \cdots < i_c \leq s} H_{i_1} \cap \cdots \cap H_{i_c}.$$

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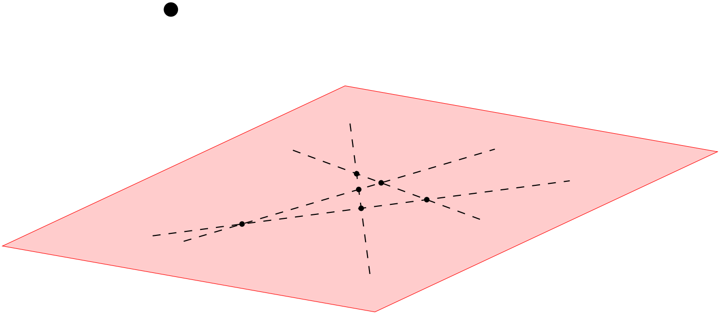
We relax the definition so that the intersection of $j \geq 4$ hyperplanes in $\mathbb{P}^3$ may equal a point and call these new configurations *pseudostars*.

Example

In which four planes meet at a point.

An easy example of a pseudostar is a projective cone over a star in $\mathbb{P}^2$.



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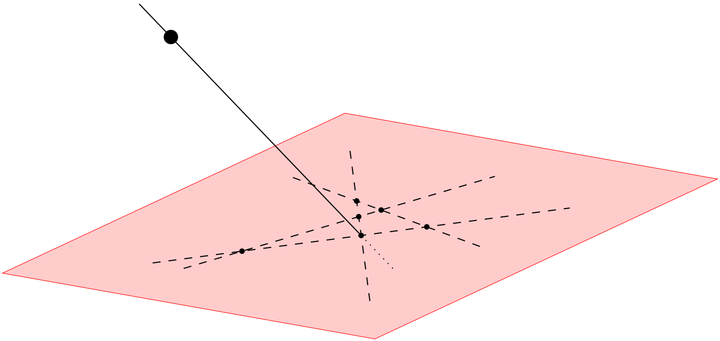

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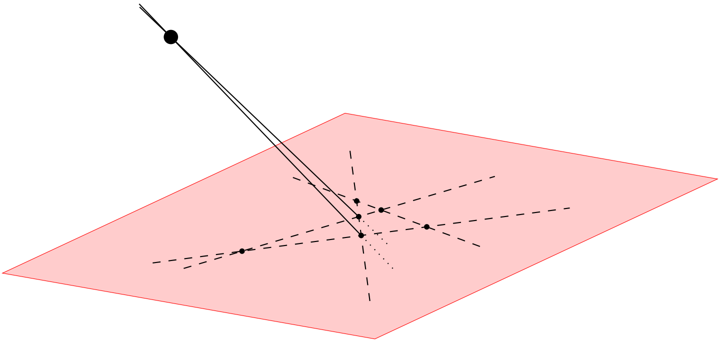

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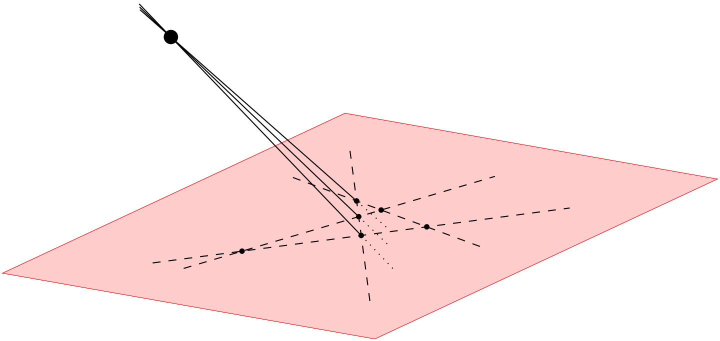

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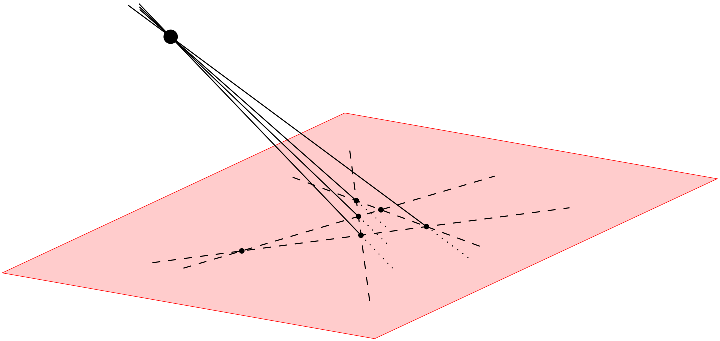

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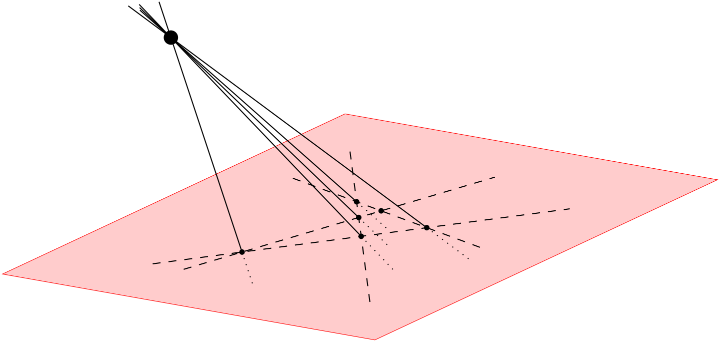

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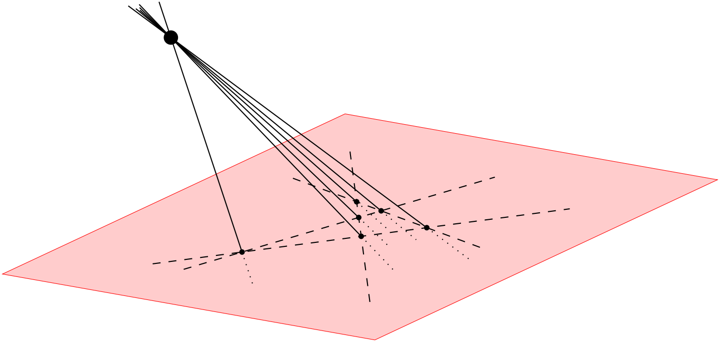

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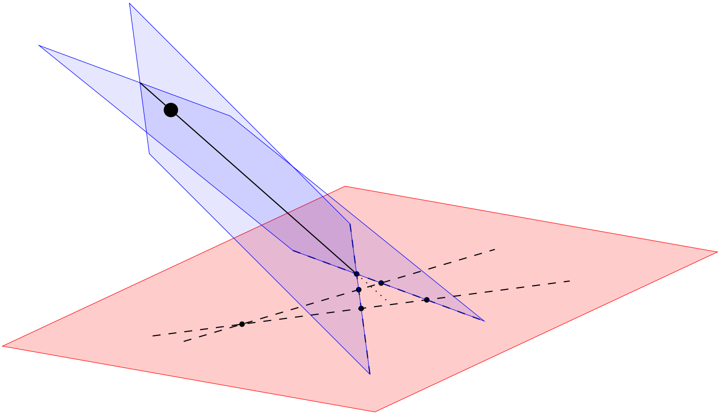

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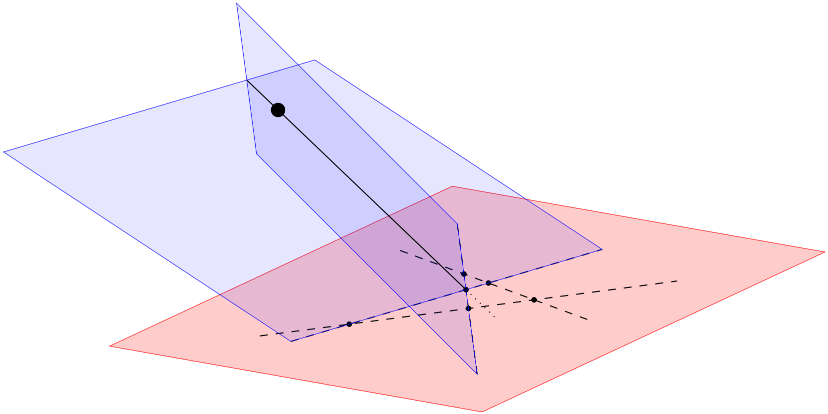

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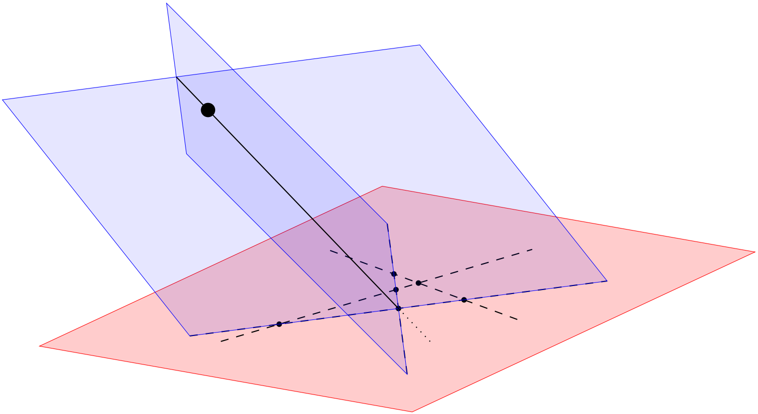

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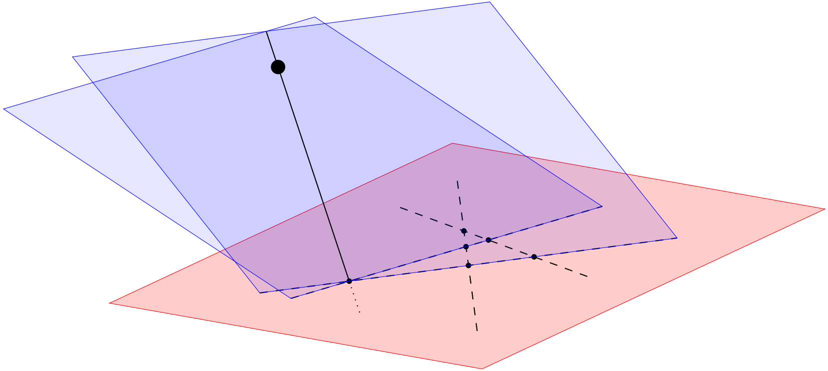

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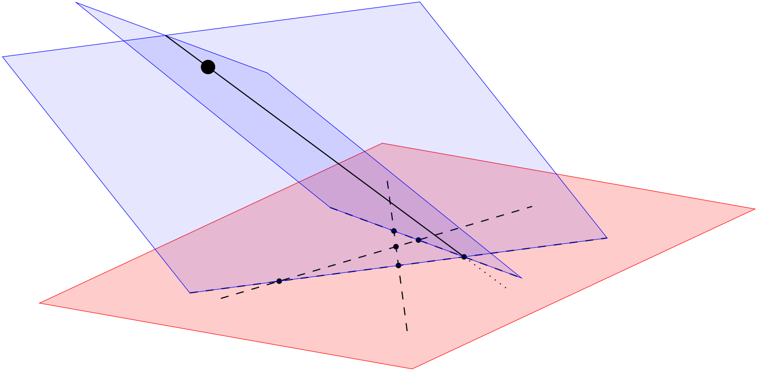

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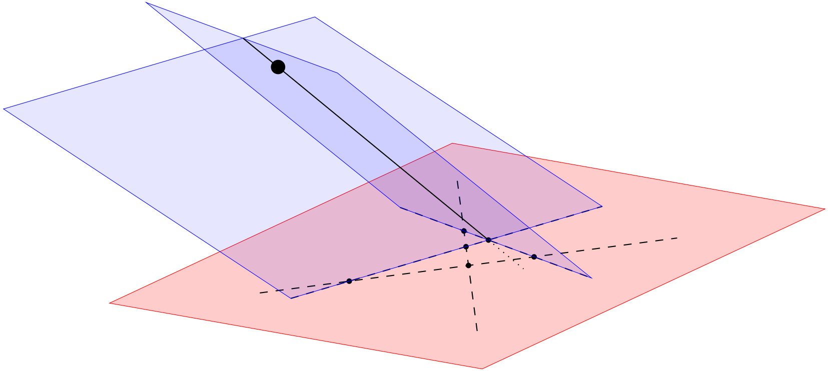

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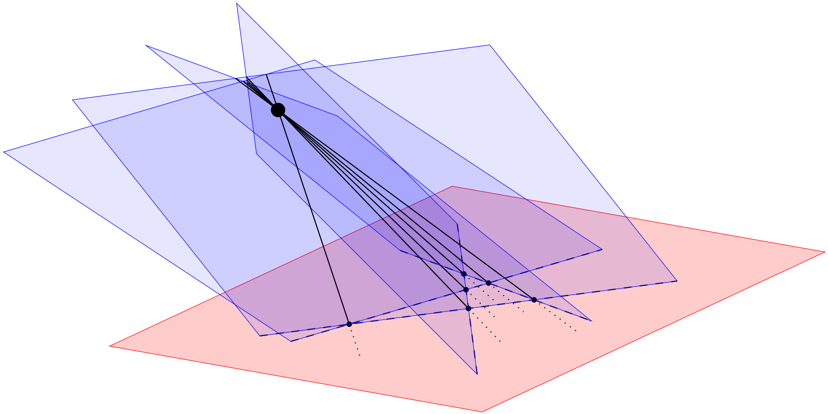

ACMness

We note the following useful property of ACM subschemes of positive dimension:

Proposition

Let $X \subseteq \mathbb{P}^N$ be an arithmetically Cohen-Macaulay scheme of dimension at least 1, and suppose $H \subseteq \mathbb{P}^N$ is a general hyperplane. Let $X \cap H$ denote the general hyperplane section of X, $S = k[\mathbb{P}^N]$, and $R = S/I(H) \cong k[\mathbb{P}^{N-1}]$. Then the Hilbert function of $R/I(X \cap H)$ is given by

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Consequence: if $X \subseteq \mathbb{P}^N$ is ACM of dimension at least 1, and H is a general hyperplane in $\mathbb{P}^N$, then $\alpha(X \cap H) = \alpha(X)$.

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- *Pseudostars* and their symbolic powers are ACM.

α for pseudostars and general hyperplane sections

Lemma

Let $\mathbb{L}$ be a pseudo-star in $\mathbb{P}^3$ formed by the pairwise intersections of $d \geq 3$ planes $\{H_1, \ldots, H_d\}$. Then $\alpha(\mathbb{L}) = d - 1$ and $\alpha(2\mathbb{L}) = d$. *

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Lemma

If $d \geq 3$ lines in $\mathbb{P}^3$ meet a general hyperplane in collinear points, the lines are coplanar. **

Characterization

Theorem (–)

Let $\mathbb{L} \subseteq \mathbb{P}^3$ be a union of lines $\ell_1, \ell_2, \ldots, \ell_s$. TFAE:

(a) $\mathbb{L}$ *is ACM of type with* $d = \alpha(2\mathbb{L}) = \alpha(\mathbb{L}) + 1$ *for some* $d > 1$.

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Sketch of (b) $\Rightarrow$ (a): If $\mathbb{L}$ is coplanar, $\alpha(\mathbb{L}) = 1$ and $\alpha(2\mathbb{L}) = 2$. If $\mathbb{L}$ is a pseudostar formed by the pairwise intersection of d planes, then $\alpha(2\mathbb{L}) = d$ and $\alpha(\mathbb{L}) = d - 1$ by previous slides.

Characterization

Sketch of (a) $\Rightarrow$ (b): Since α is preserved by general hyperplane sections, we have $\alpha(2(\mathbb{L} \cap H)) = d$ and $\alpha(\mathbb{L} \cap H) = d - 1$, so is a star or collinear:

- If $\mathbb{L} \cap H$ is collinear, $\mathbb{L}$ is coplanar.
- If a $\mathbb{L} \cap H$ is a star, consider an example.

Proof Sketch via Star

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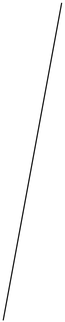

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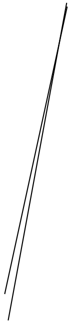

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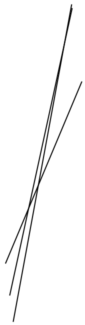

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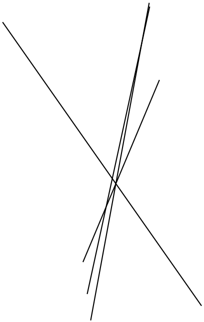

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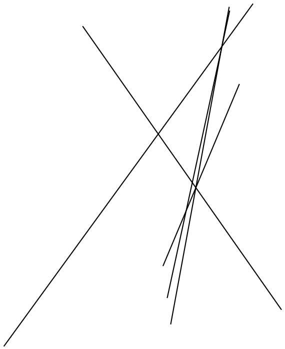

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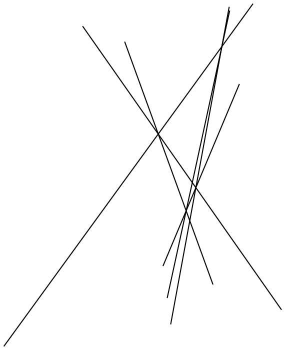

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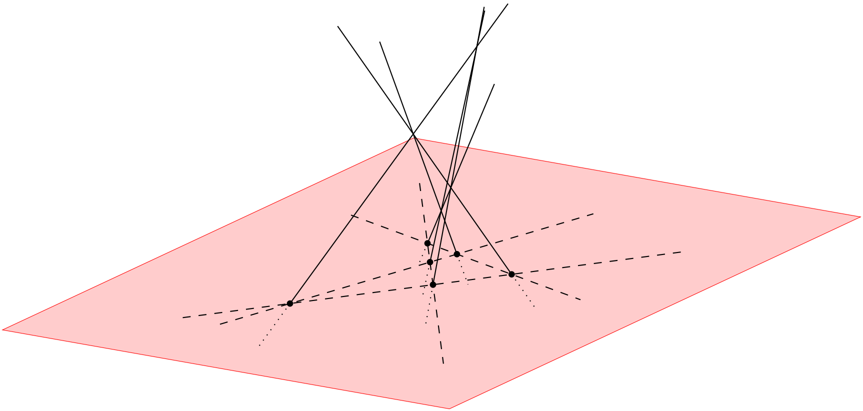

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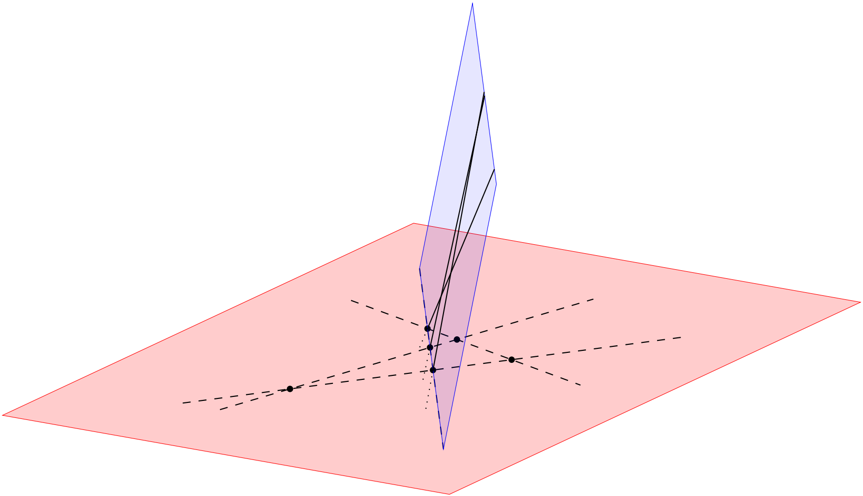

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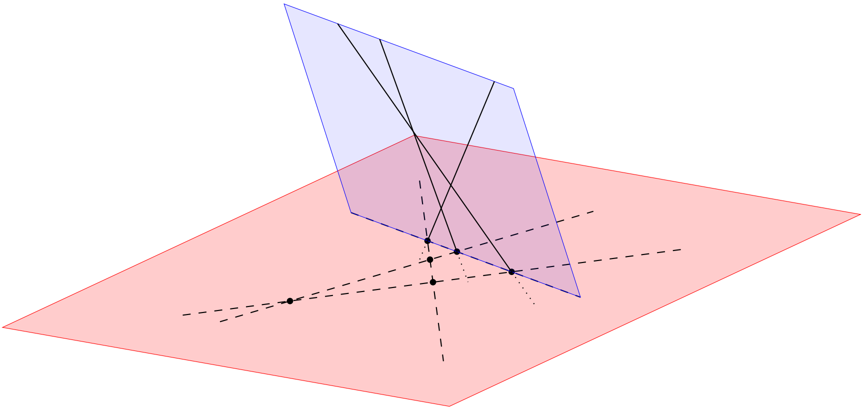

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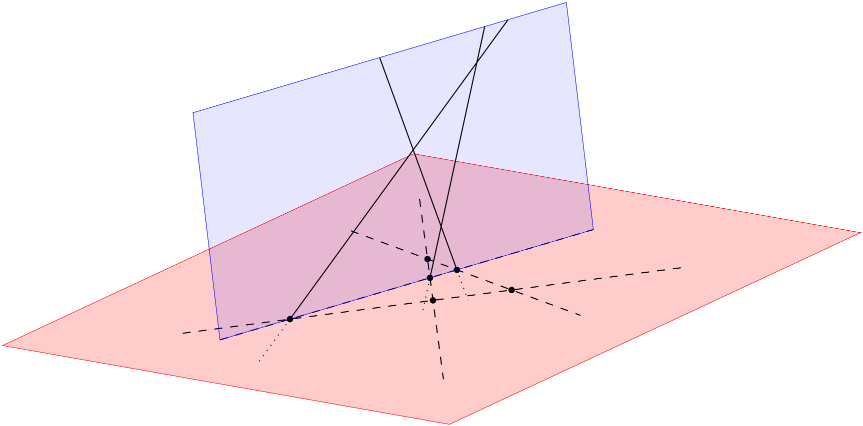

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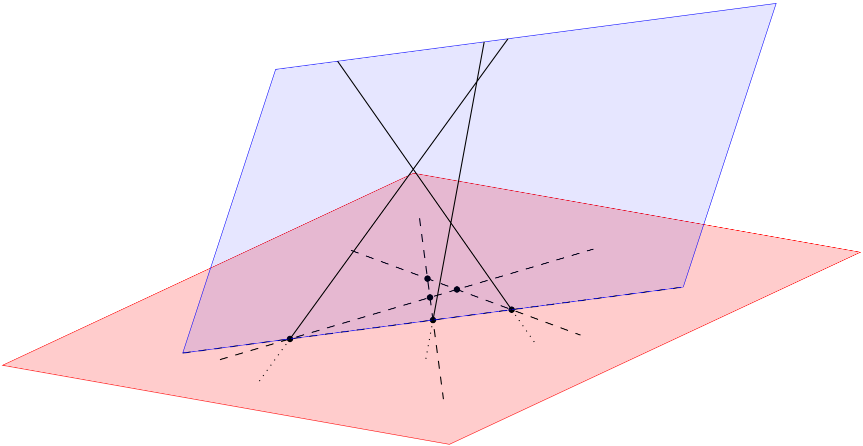

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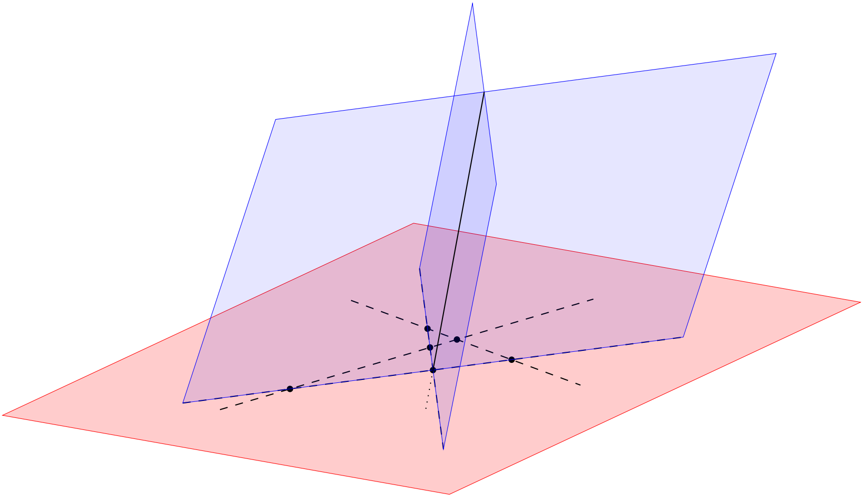

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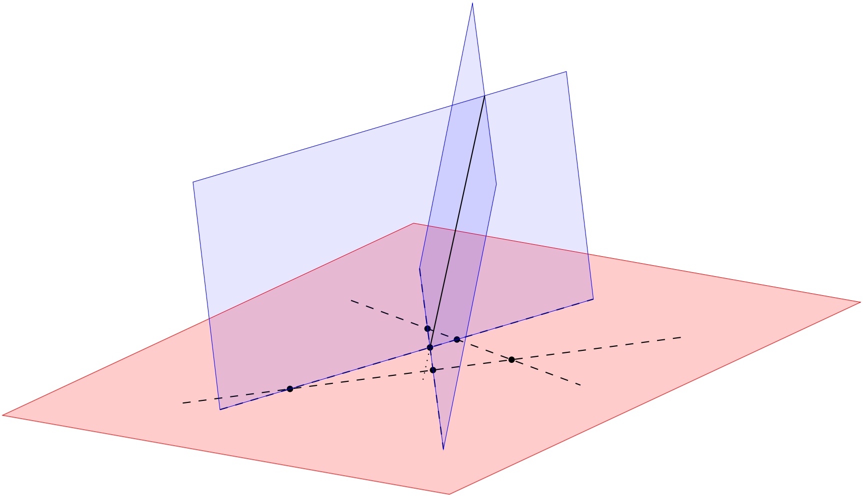

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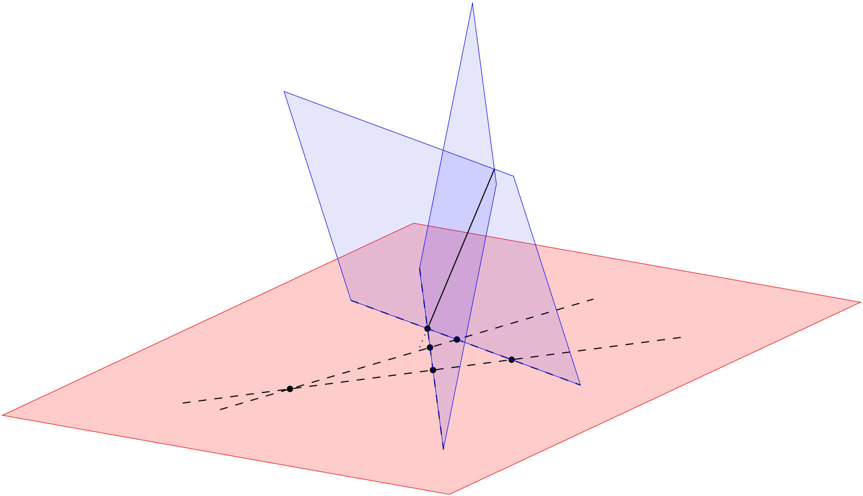

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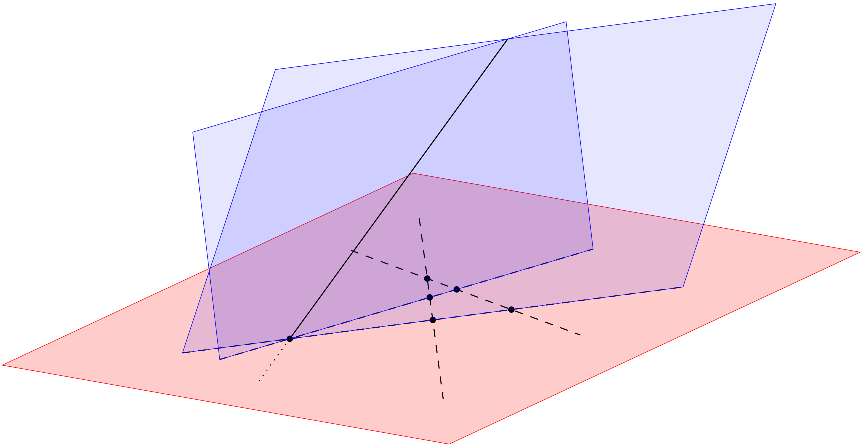

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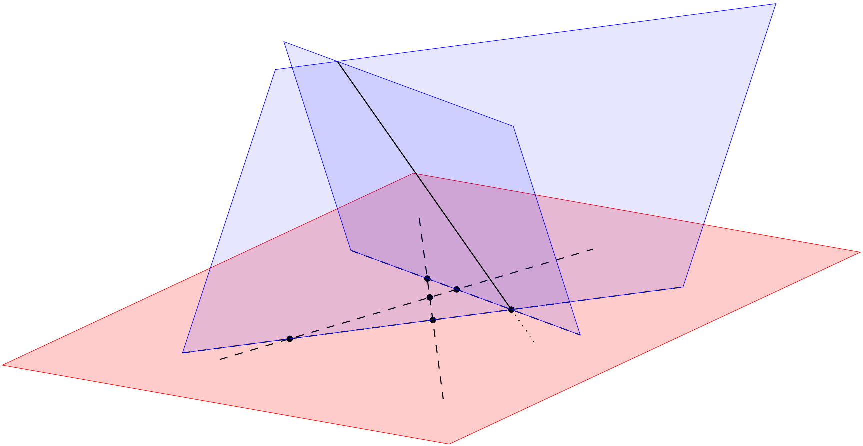

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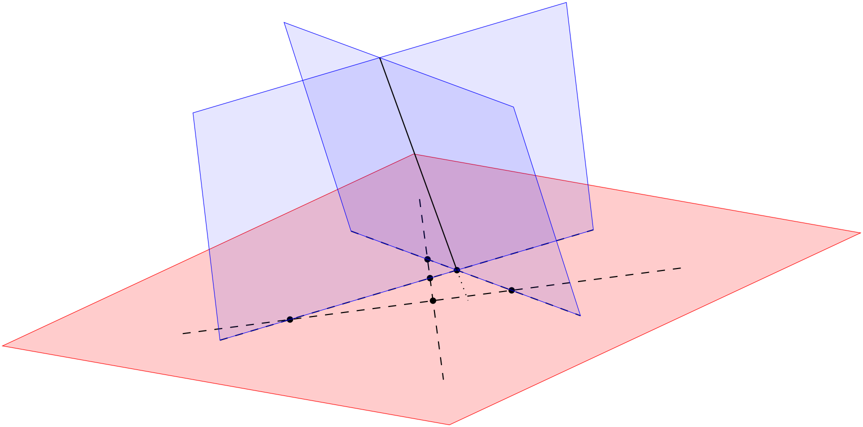

Proof Sketch via Star

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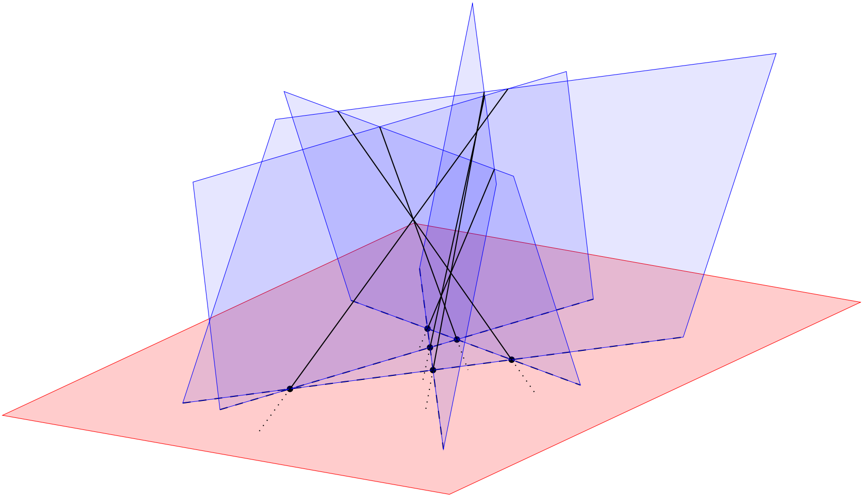

Questions

Question

Does there exist a non-ACM configuration of lines $\mathbb{L}$ for which $\alpha(2\mathbb{L}) = \alpha(\mathbb{L}) + 1$?

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Does there exist a non-ACM configuration of lines $\mathbb{L}$ *for which* $\alpha(2\mathbb{L}) = \alpha(\mathbb{L}) + 1$*?*

Question

Which configurations of lines in $\mathbb{P}^3$ *are ACM?*

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Which configurations of lines in $\mathbb{P}^3$ are ACM?

Question

Which reduced (possibly irreducible) curves C in $\mathbb{P}^3$ have type $\alpha(2C) = \alpha(C) + 1$?

Thanks

Thank you!