THE IMPORTANCE OF α

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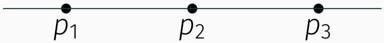
BACKGROUND

Given a configuration of points in the projective plane $\mathbb{P}^2$, what is the least degree of a polynomial which vanishes at all the points?

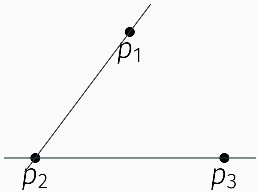
Well, it depends.

two points

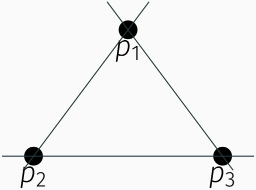
AN EXAMPLE



A DIFFERENT CONFIGURATION



WHAT ABOUT DOUBLE VANISHING?



Question

What can we learn about the arrangements of points in the plane (or lines in space) if we know the corresponding increase in degree from single to double vanishing is small?

Our setup:

- Let k be an algebraically closed field (e.g., $k = \mathbb{C}$).
- We will first consider (reduced) points $p_1, p_2, \ldots, p_r \in \mathbb{P}^2_k$.
- Let $R = k[\mathbb{P}^2] = k[x, y, z]$ be the homogeneous coordinate ring of $\mathbb{P}^2$.
- By $I(p_j)$ we mean the (homogeneous) ideal generated by all homogeneous polynomials (forms) F such that $F(p_j) = 0$. The ideal $I = \bigcap_{j=1}^{r} I(p_j)$ is the ideal generated by all forms vanishing at all points p_j.
- The m-th symbolic power of I, denoted $I^{(m)}$ is the ideal generated by all forms vanishing to order at least m at all points p_j:

$$I^{(m)} = \bigcap_{j=1}^{r} I(p_j)^m.$$

Definition

Given a nontrivial homogeneous ideal $J \subseteq k[x, y, z]$, we let $\alpha(J)$ denote the minimum degree of a nonzero element of J.

Thus, if $I = \cap I(p_j)$, $\alpha(I^{(m)})$ is essentially the least degree of a polynomial vanishing at each point p_j to order at least m.

Definition

Given a nontrivial homogeneous ideal I, the initial sequence (or α-sequence) is the sequence $\left(\alpha(I^{(m)})\right)_{m\geq 1}$.

Fact: We have $\alpha(I) < \alpha(I^{(2)}) < \alpha(I^{(3)}) < \cdots$.

Remark: The *Waldschmidt constant* $\hat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}$ is an asymptotic version of α which has also been the focus of recent study.

Question (Bocci-Chiantini 2011)

Let $Z \subseteq \mathbb{P}^2$ be a finite set of points and $I = I(Z)$. What role does the difference $\alpha(I^{(2)}) - \alpha(I)$ play in determining the geometry of Z?

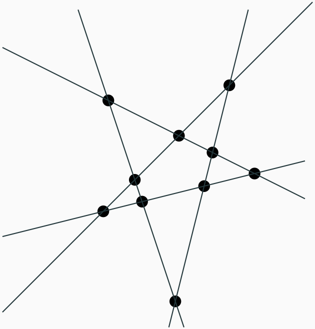
RESULTS IN ℓ^2

Theorem

Let $Z \subseteq \mathbb{P}^2$ be a finite set of points and suppose $I = I(Z)$. Then $\alpha(I^{(2)}) = \alpha(I) + 1$ if and only if Z is collinear or a star configuration.

Definition

The set $Z \subseteq \mathbb{P}^2$ of $\binom{d}{2}$ points obtained by taking pairwise intersections of d distinct lines such that no three lines meet in a point is called a star configuration.



$$\alpha(I(Z)) \leqslant 4$$

$$\alpha(I(Z)^{(2)}) \leqslant 5$$

RESULTS IN $\mathbb{R}^3$

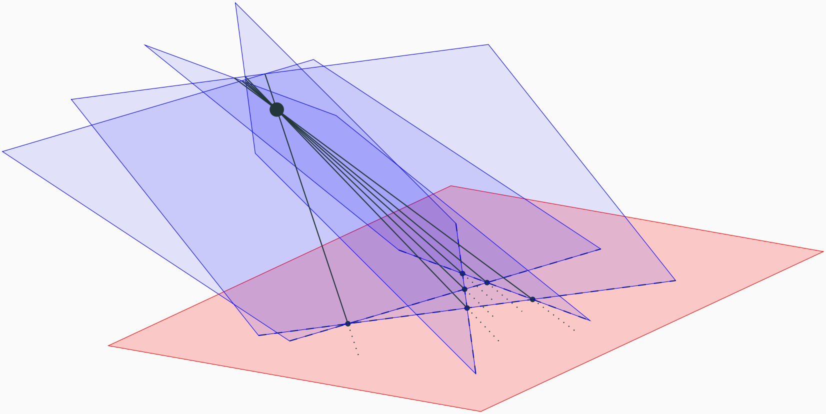
Question

What can we say about codimension 2 objects in higher dimensional projective spaces?

Definition

A pseudo-star configuration of lines in $\mathbb{P}^3$ is a finite collection of lines formed by the pairwise intersections of hyperplanes such that no three of the hyperplanes meet in a line.

An easy example of a pseudostar is a projective cone over a star in $\mathbb{P}^2$.



Theorem (–)

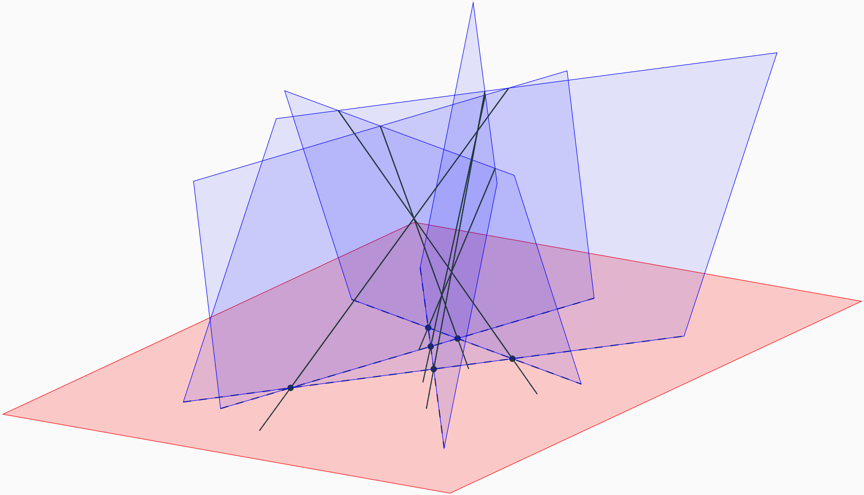
Let $\mathbb{L} \subseteq \mathbb{P}^3$ be a union of lines $\ell_1, \ell_2, \ldots, \ell_r$ in $\mathbb{P}^3$ and $I = I(\mathbb{L})$. TFAE:

(a) *$\mathbb{L}$ is ACM with $d = \alpha(I^{(2)}) = \alpha(I) + 1$ for some $d > 1$.*

(b) *$\mathbb{L}$ is either a pseudostar or coplanar.*

Idea: If $X \subseteq \mathbb{P}^3$ is ACM then $\alpha(I(X)) = \alpha(I(X \cap H))$.

PROOF SKETCH VIA STAR



Conjecture

Let $C \subseteq \mathbb{P}^3$ be a reduced, unmixed curve with $I = I(C)$. If

$$\alpha(I^{(2)}) = \alpha(I) + 1 = d \tag{1}$$

then C is ACM and is either a plane curve or a pseudostar configuration defined by d hyperplanes. In particular, (1) never holds for a non-ACM curve.

THANKS

Thank you!