ON SYMBOLIC POWERS OF IDEALS

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Exploring Symbolic Powers

Definition

Let I be an ideal in a Noetherian ring R, and $m \geq 1$. Then the **m-th symbolic power of I**, denoted $I^{(m)}$, is the ideal

$$I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} (I^m R_P \cap R),$$

where R_P denotes the localization of R at the prime ideal P.

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Theorem
Let I be a radical ideal in a Noetherian ring R with minimal primes $P_1, P_2, \ldots, P_s$. Then $I = P_1 \cap P_2 \cap \cdots \cap P_s$, and

$$I^{(m)} = P_1^{(m)} \cap P_2^{(m)} \cap \cdots \cap P_s^{(m)}.$$

Theorem
Let R be Noetherian and suppose $I \subseteq R$ is an ideal generated by a regular sequence. Then $I^{(m)} = I^m$ for all $m \geq 1$.

THEOREM AND AN EXAMPLE

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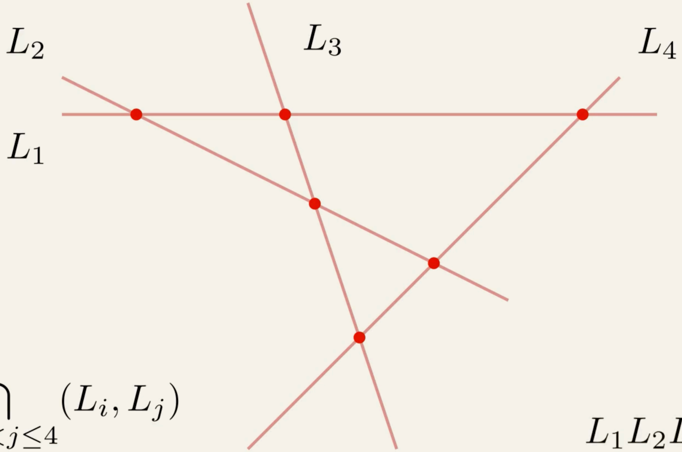
Example
Let $R = k[\mathbb{P}^2] = k[x, y, z]$ and $p \in \mathbb{P}^2$. Then $I = I(p)$ can be taken to be $I = (x, y)$, and

$$I^{(m)} = (x, y)^{(m)} = (x, y)^m.$$

Theorem (Zariski, Nagata)
Let k be a perfect field, $R = k[x_0, x_1, \ldots, x_N]$, $I \subseteq R$ a radical ideal, and $X \subseteq \mathbb{P}^N$ the variety corresponding to I. Then $I^{(m)}$ is the ideal generated by forms vanishing to order at least m on X.

Example: Star Configuration in $\mathbb{P}^2$



$$I = \bigcap_{1 \le i < j \le 4} (L_i, L_j)$$

$$L_1 L_2 L_3 L_4 \in I^{(2)}$$

TWO CONTEXTS

- Ideals of (fat) points
- Squarefree monomial ideals

The Containment Problem and Ideals of Points

OUR QUESTION (FIRST DRAFT)

Question
Given a nontrivial homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$*, how do* $I^{(m)}$ *and* I^r *compare?*

Theorem
Let I be an ideal in a Noetherian ring R. Then:

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- *if* R *is a domain,* $I^m \subseteq I^{(r)}$ *if and only if* $m \geq r$.
- $I^{(m)} \subseteq I^r$ *implies* $m \geq r$*, but the converse need not hold.*

OUR (GENERAL) QUESTION (FINAL DRAFT)

Containment Problem: Given a nontrivial homogeneous ideal $I \subseteq k[x_0, x_1, x_2, \ldots, x_N]$, for which m, r do we have $I^{(m)} \subseteq I^r$?

Theorem (Ein-Lazarsfeld-Smith (2001), Hochster-Huneke (2002), Ma-Schwede (2017), Murayama (2021))
Let R be a regular ring and I a radical ideal in R of big height e. Then if $m \geq er$, $I^{(m)} \subseteq I^r$.

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Corollary
Let I be a nontrivial homogeneous ideal in $k[\mathbb{P}^N]$. If $m \geq Nr$, then $I^{(m)} \subseteq I^r$.

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Corollary
Let I be a nontrivial homogeneous ideal in $k[\mathbb{P}^N]$. If $m \geq Nr$, then $I^{(m)} \subseteq I^r$.

Question (Huneke)
When $I = I(S)$ is the ideal defining any finite set S of points in $\mathbb{P}^2$, is it true that $I^{(3)} \subseteq I^2$?

Definition
If $p_i \in \mathbb{P}^N$ and $Z = m_1p_1 + m_2p_2 + \cdots m_sp_s$ is a **fat points subscheme** with $I = I(Z)$, then

$$I(Z) = I(p_1)^{m_1} \cap I(p_2)^{m_2} \cap \cdots \cap I(p_s)^{m_s}.$$

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The symbolic powers of $I = I(Z)$ are therefore

$$I^{(m)} = I(mZ) = I(p_1)^{mm_1} \cap I(p_2)^{mm_2} \cap \cdots \cap I(p_s)^{mm_s}.$$

- Answered Huneke's question in the affirmative for $I(S)$ when S is a finite set of generic points in $\mathbb{P}^2$.

COMPARING POWERS AND SYMBOLIC POWERS OF IDEALS (2010; WITH C. BOCCI)

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- Introduced *the resurgence*, $\rho(I)$, the supremum of the ratios m/r for which $I^{(m)} \not\subseteq I^r$, and calculated ρ for ideals of various point configurations in $\mathbb{P}^2$.

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- Obtained bounds on $\rho(I(Z))$ in terms of other invariants of $I(Z)$.
- Used these bounds to establish the sharpness of the uniform bound.

THE RESURGENCE OF IDEALS OF POINTS AND THE CONTAINMENT PROBLEM (2010; WITH C. BOCCI)

Theorem
Assume the points $p_1, \ldots, p_n$ lie on a smooth conic curve. Let $I = I(Z)$ where $Z = p_1 + \cdots + p_n$. Let $m, r > 0$.

1. *If n is even or $n = 1$, then $I^{(m)} \subseteq I^r$ if and only if $m \geq r$. In particular, $\rho(I) = 1$.*
2. *If $n > 1$ is odd, then $I^{(m)} \subseteq I^r$ if and only if $(n+1)r - 1 \leq nm$; in particular, $\rho(I) = (n+1)/n$.*

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Conjecture (B. Harbourne)
Let $I \subseteq k[\mathbb{P}^N]$ be a homogeneous ideal. Then $I^{(m)} \subseteq I^r$ if $m \geq rN - (N-1)$.

Squarefree Monomial Ideals

Oberwolfach Mini-Workshop: Ideals of Linear Subspaces,
Their Symbolic Powers and Waring Problems (2015)

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Definition
The Waldschmidt constant, denoted $\widehat{\alpha}(I)$, is the limit

$$\widehat{\alpha}(I) := \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

Example
Let $R = k[x, y, z]$ and set $I = (xy, yz, xz) = (x, y) \cap (x, z) \cap (y, z)$. It turns out that

$$I^{(m)} = (x, y)^m \cap (x, z)^m \cap (y, z)^m.$$

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Given $I = (x, y) \cap (x, z) \cap (y, z) \subseteq k[x, y, z]$:

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Example

Given $I = (x, y) \cap (x, z) \cap (y, z) \subseteq k[x, y, z]$:

$$\begin{aligned} \alpha(I)/1 &= 2/1 \\ \alpha(I^{(2)})/2 &= 3/2 \\ \alpha(I^{(3)})/3 &= 5/3 \end{aligned}$$

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Let $R = \Bbbk[x, y, z]$ and set $I = (xy, yz, xz) = (x, y) \cap (x, z) \cap (y, z)$. It turns out that

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Given $I = (x, y) \cap (x, z) \cap (y, z) \subseteq k[x, y, z]$:

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In fact, $\widehat{\alpha}(I) = \frac{3}{2}$.

Theorem
Let I be a squarefree monomial ideal in $k[x_1, \ldots, x_N]$.

1. *There exist unique prime ideals of the form $P_i = (x_{i,1}, \ldots, x_{i,t_i})$ such that $I = P_1 \cap \cdots \cap P_s$.*

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2. *With the P_i's as above, we have*

$$I^{(m)} = P_1^m \cap \cdots \cap P_s^m.$$

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3. *For all $m \geq 1$,*

$$\alpha(I^{(m)}) = \min\{a_1 + \cdots + a_N \mid x_1^{a_1} \cdots x_N^{a_N} \in I^{(m)}\}.$$

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$$\alpha(I^{(m)}) = \min\{a_1 + \cdots + a_N \mid x_1^{a_1} \cdots x_N^{a_N} \in I^{(m)}\}.$$

We therefore have $x_1^{a_1} \cdots x_N^{a_N} \in I^{(m)}$ *if and only if* $a_{i,1} + \cdots + a_{i,t_i} \geq m$ *for* $i = 1, \ldots, s$.

Example

Let $I = (x_1x_3x_5, x_2x_3x_4, x_1x_2x_4x_5, x_3x_4x_5) \subseteq k[x_1, x_2, \ldots, x_5]$. Then

$$I^{(m)} = (x_1, x_3)^m \cap (x_2, x_3)^m \cap (x_1, x_4)^m \cap (x_3, x_4)^m \cap (x_2, x_5)^m \cap (x_3, x_5)^m \cap (x_4, x_5)^m.$$

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Determining if $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5} \in I^{(m)}$ is equivalent to determining if the following system of inequalities are satisfied:

$$a_1 + a_3 \geq m \leftrightarrow x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5} \in (x_1, x_3)^m$$
$$a_2 + a_3 \geq m \leftrightarrow x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5} \in (x_2, x_3)^m$$
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To calculate $\alpha(I^{(m)})$, we wish to minimize $a_1 + a_2 + a_3 + a_4 + a_5$ subject to the above constraints.

Theorem (Bocci et al. (2016))
Let $I \subseteq k[x_1, \ldots, x_N]$ be a squarefree monomial ideal with minimal primary decomposition $I = P_1 \cap \cdots \cap P_s$ with $P_i = (x_{i,1}, \ldots, x_{i,t_i})$ for $i = 1, \ldots, s$. Let A be the $s \times n$ matrix where

$$A_{i,j} = \begin{cases} 1 & \text{if } x_j \in P_i \\ 0 & \text{if } x_j \notin P_i. \end{cases}$$

Consider the following linear program (LP):

$$\text{minimize } \mathbf{1}^T \mathbf{y}$$

$$\text{subject to } A\mathbf{y} \geq \mathbf{1} \text{ and } \mathbf{y} \geq \mathbf{0}$$

and suppose $\mathbf{y}^*$ is a feasible solution that realizes the optimal value. Then

$$\widehat{\alpha}(I) = \mathbf{1}^T \mathbf{y}^*.$$

That is, $\widehat{\alpha}(I)$ is the optimal value of the LP.

Application to Edge Ideals

Definition
Let G be a (finite, simple) graph with vertices $x_1, x_2, \ldots, x_N$. The **edge ideal** $I(G)$ is the ideal in $k[x_1, \ldots, x_N]$ generated by the set

$$\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}.$$

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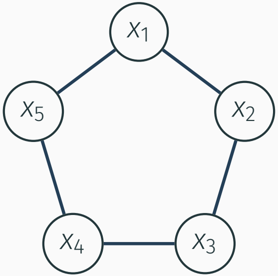
When $I = I(G)$, the minimal primes of I are generated by the variables corresponding to the minimal vertex covers of G.

EXAMPLE: $I(C_5)$



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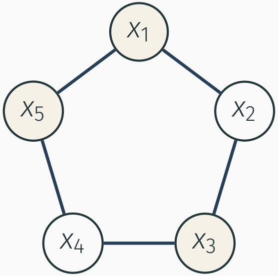

Minimal vertex covers:

- $W_1 = \{x_1, x_3, x_5\}$

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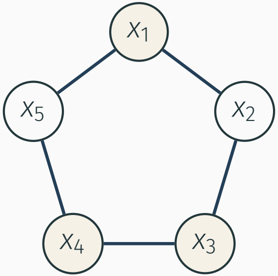

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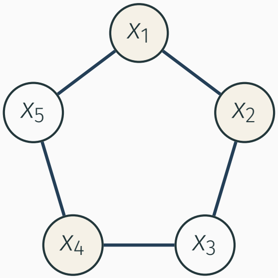

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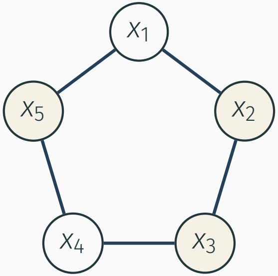

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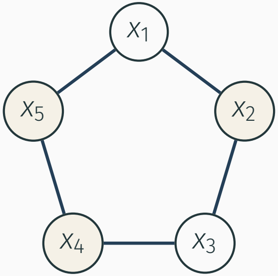

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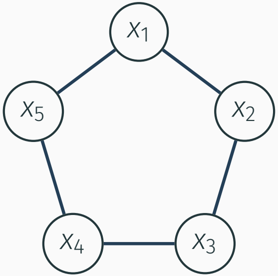

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Thus,

$$I(C_5)^{(m)} = (x_1, x_3, x_5)^m \cap (x_1, x_3, x_4)^m \cap (x_1, x_2, x_4)^m \cap (x_2, x_3, x_5)^m \cap (x_2, x_4, x_5)^m .$$

Theorem (Bocci et al. (2016))
Let G be a finite simple graph with edge ideal $I(G)$. Then

$$\widehat{\alpha}(I(G)) = \frac{\chi_f(G)}{\chi_f(G) - 1},$$

where $\chi_f(G)$ denotes the fractional chromatic number of G.

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Theorem (Bocci et al. (2016))
Let G be a nonempty graph.

1. *If $\chi(G) = \omega(G)$, then $\widehat{\alpha}(I(G)) = \frac{\chi(G)}{\chi(G)-1}$.*
2. *If G is k-partite, then $\widehat{\alpha}(I(G)) \geq \frac{k}{k-1}$. When G is complete k-partite, $\widehat{\alpha}(I(G)) = \frac{k}{k-1}$.*
3. *If G is bipartite, $\widehat{\alpha}(I(G)) = 2$.*
4. *If $G = C_{2n+1}$ is an odd cycle, then $\widehat{\alpha}(I(C_{2n+1})) = \frac{2n+1}{n+1}$.*
5. *If $G = C^c_{2n+1}$, then $\widehat{\alpha}(I(C^c_{2n+1})) = \frac{2n+1}{2n-1}$.*

Theorem (J–, Kamp, and Vander Woude (2019))
Let I be the edge ideal of an odd cycle on $2n+1$ vertices. Then:

1. $I^{(m)} = I^m$ *for* $1 \leq m \leq n$.

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3. $\rho(I) = \frac{2n+2}{2n+1}$.

RESOURCES

- *Symbolic Powers of Ideals* (2018), by Dao et al.

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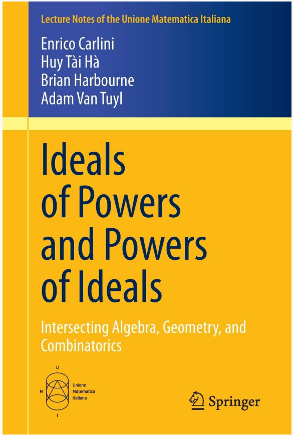

Thanks!