EXPLORING ALGEBRA VIA GRAPHS

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Algebra $\leftrightarrow$ Geometry

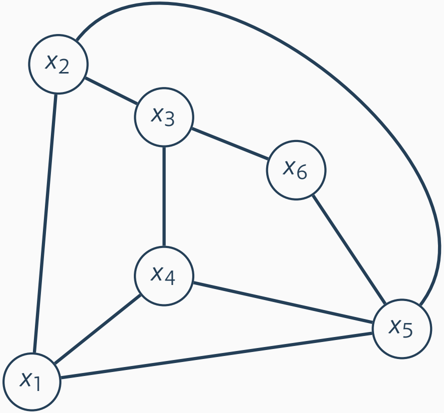
Example: Given a quadratic polynomial $ax^2 + bx + c$, what does $b^2 - 4ac$ tell us?

Algebraic structure $\leftrightarrow$ Graph

Graph Theory

WHAT IS A GRAPH?

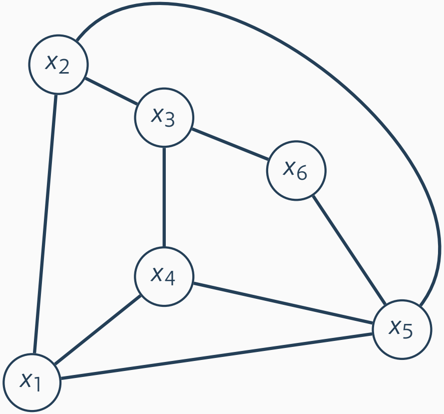
Graphs describe CONNECTIONS.



FORMALLY

DEFINITION

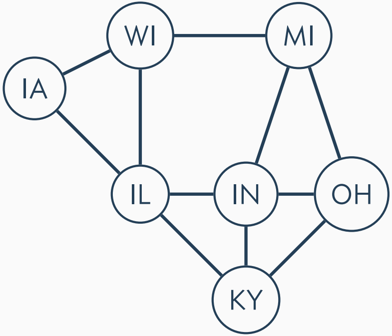
*A **graph** G is a pair of sets $G = (V, E)$ where $V = \{x_1, x_2, \ldots, x_n\}$ consists of **vertices**. The set E of **edges** consists of two-element subsets of V.*



$$V = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

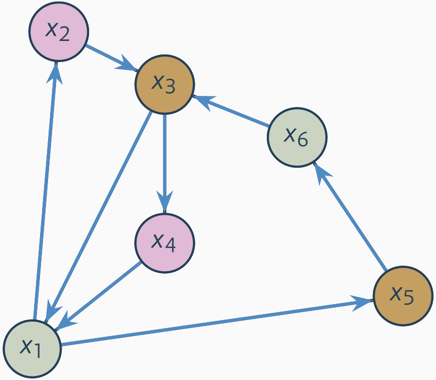
$$E = \{\{x_1, x_2\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_3\}, \{x_2, x_5\}, \{x_3, x_4\}, \{x_3, x_6\}, \{x_4, x_5\}, \{x_5, x_6\}\}$$

EXAMPLE: MAPS



SOME BIG QUESTIONS

- Is it possible to walk from vertex to vertex, cross each edge exactly once, and end where you began?
- Is it possible to walk along the graph so that you visit each vertex exactly once, and end where you began?
- What is the minimum number of colors required to color each vertex so that ADJACENT vertices are assigned different colors?

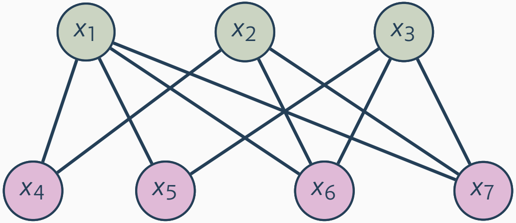


THE CHROMATIC NUMBER

Definition

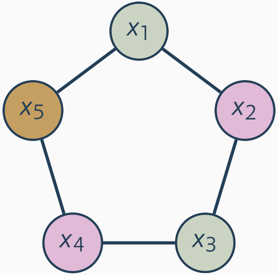
*The **chromatic number** of a graph G, denoted $\chi(G)$, is the minimum number of colors required to **properly color** the vertices of G.*

EXAMPLE: BIPARTITE GRAPHS



$$\chi(G) = 2$$

EXAMPLE: ODD CYCLE, C_5



$$\chi(G) = 3$$

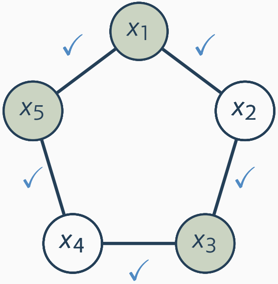
Definition

*A **vertex cover** of a graph $G = (V, E)$ is a set $W \subseteq V$ such that every edge meets W. We say W is **minimal** if no set $K \subsetneq W$ is a vertex cover.*

Minimal Vertex Cover Problem: Given a graph G, find its minimal vertex covers.

EXAMPLE: ODD CYCLE

GOAL: Find a subset W of V which covers every edge.



So, A minimal vertex cover of C_5 is $W = \{x_1, x_3, x_5\}$.

Algebraic Structure

ABSTRACTION OF ALGEBRA

- Historically: find solutions to polynomial equations like $f(x) = 0$
- Remained central to algebra into the 19th century
- Led to increasing abstraction; fields implicit in Galois' work in 1830
- Dedekind developed the notion of an IDEAL NUMBER
- By the early 20th century, Emmy Noether gave the modern definition of RING and IDEAL

POLYNOMIAL RINGS AND IDEALS

For us, a **RING** $(R, +, \cdot)$ is a set closed under two operations, addition and multiplication, that behave like the integers, $\mathbb{Z}$.

Example

The set of all polynomials with real coefficients, denoted $\mathbb{R}[x]$, is also a ring. So is $\mathbb{R}[x, y]$, which contains polynomials like $y^3 - \sqrt{2}y^2x + xy - 2$.

Definition

An ***ideal*** *I is a nonempty subset of a ring R for which there exists a (finite) set $S = \{s_1, s_2, \ldots, s_n\}$ such that any $f \in I$ can be written as an R-"linear combination" of elements of S. That is:*

$$f = r_1 s_1 + r_2 s_2 + \cdots + r_n s_n$$

We often write $I = \langle s_1, s_2, \ldots, s_n \rangle$.

Analogy: vector subspaces have bases!

EXAMPLES OF IDEALS

Example

In $\mathbb{Z}$, we have ideals such as:

- $\langle 3 \rangle = \{3x \,:\, x \in \mathbb{Z}\}$
- $\langle 10, 15 \rangle = \{10x + 15y \,:\, x, y \in \mathbb{Z}\}$
- $\langle 17, 41 \rangle = \{17x + 41y \,:\, x, y \in \mathbb{Z}\}$

Is $10 \in \langle 17, 41 \rangle$? Yes! Since

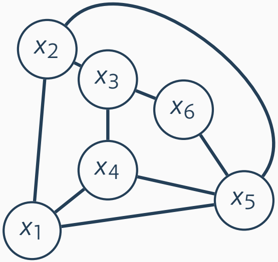
$$10 = (-120) \cdot 17 + 50 \cdot 41.$$

IMPORTANT EXAMPLE

Example

Consider the multivariate polynomial ring $R = \mathbb{R}[x_1, x_2, x_3, x_4, x_5, x_6]$ and the ideal

$$I = \langle x_1x_2, x_1x_4, x_1x_5, x_2x_5, x_4x_5, x_2x_3, x_3x_4, x_3x_6, x_5x_6 \rangle.$$



THE EDGE IDEAL CORRESPONDENCE

Definition (Simis-Vasconcelos-Villareal (1994))

Let $G = (V, E)$ be a graph on the vertex set $V = \{x_1, x_2, \dots, x_n\}$ and let $R = \mathbb{R}[x_1, x_2, \dots, x_n]$. The ***edge ideal*** *of G is the ideal generated by products of pairs of variables corresponding to edges in G:*

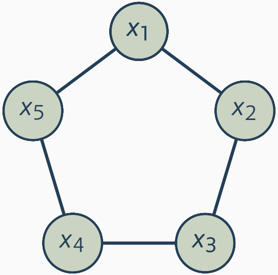
$$I(G) = \left\langle x_i x_j \mid \left\{x_i, x_j\right\} \in E \right\rangle.$$

So, each edge ideal specifies a graph and vice versa.

Theorem

Let $I = I(G)$ and let $W_1, W_2, \ldots, W_k$ be the minimal vertex covers of G. Then $I = \langle W_1 \rangle \cap \langle W_2 \rangle \cap \cdots \cap \langle W_k \rangle$.

EXAMPLE: C_5



- $W_1 = \{x_1, x_3, x_5\}$
- $W_2 = \{x_1, x_3, x_4\}$
- $W_3 = \{x_1, x_2, x_4\}$
- $W_4 = \{x_2, x_3, x_5\}$
- $W_5 = \{x_2, x_4, x_5\}$

Thus,

$$I(C_5) = \langle x_1, x_3, x_5 \rangle \cap \langle x_1, x_3, x_4 \rangle \cap \langle x_1, x_2, x_4 \rangle \cap \langle x_2, x_3, x_5 \rangle \cap \langle x_2, x_4, x_5 \rangle.$$

Definition

*Given an edge ideal $I(G) = \langle W_1 \rangle \cap \langle W_2 \rangle \cap \cdots \cap \langle W_k \rangle$, the **$m$-th symbolic power** of $I(G)$ is*

$$I(G)^{(m)} = \langle W_1 \rangle^m \cap \langle W_2 \rangle^m \cap \cdots \cap \langle W_k \rangle^m$$

So:

$$I(C_5) = \langle x_1, x_3, x_5 \rangle \cap \langle x_1, x_3, x_4 \rangle \cap \langle x_1, x_2, x_4 \rangle \cap \langle x_2, x_3, x_5 \rangle \cap \langle x_2, x_4, x_5 \rangle.$$

and

$$\begin{aligned} I(C_5)^{(2)} &= \left\langle x_1^2, x_1x_3, x_1x_5, x_3^2, x_3x_5, x_5^2 \right\rangle \cap \left\langle x_1^2, x_1x_3, x_1x_4, x_3^2, x_3x_4, x_4^2 \right\rangle \\ &\cap \left\langle x_1^2, x_1x_2, x_1x_4, x_2^2, x_2x_4, x_4^2 \right\rangle \cap \left\langle x_2^2, x_2x_3, x_2x_5, x_3^2, x_3x_5, x_5^2 \right\rangle \\ &\cap \left\langle x_2^2, x_2x_4, x_2x_5, x_4^2, x_4x_5, x_5^2 \right\rangle \\ &= \langle x_4^2x_5^2, x_1x_4x_5^2, x_1^2x_5^2, x_3x_4^2x_5, x_2x_3x_4x_5, x_1x_3x_4x_5, x_1x_2x_4x_5, x_1x_2x_3x_5, x_1^2x_2x_5, x_3^2x_4^2, \\ &\quad x_2x_3^2x_4, x_1x_2x_3x_4, x_2^2x_3^2, x_1x_2^2x_3, x_1^2x_2^2 \rangle \end{aligned}$$

One invariant of recent interest is the WALDSCHMIDT CONSTANT:

$$\widehat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m},$$

where $\alpha(I)$ is the least degree of a nonzero $f \in I$.

For instance, if $I = I(G)$, $\alpha(I) = 2$. Also, we just saw that $\alpha(I(C_5)^{(2)}) = 4$.

BUT: $\alpha(I(C_5)^{(3)}) = 5$.

QUESTION: If $I = I(G)$, what is $\widehat{\alpha}(I)$?

EXTENDING THE DICTIONARY

THEOREM (J–, ET. AL (2016))

If $I = I(G)$ is the edge ideal of a nonempty graph G, then

$$\frac{\chi(G)}{\chi(G)-1} \leqslant \widehat{\alpha}(I) \leqslant \frac{\omega(G)}{\omega(G)-1}.$$

THEOREM (J–, ET. AL (2016))

Let G be a nonempty graph and $I = I(G)$.

(i) *If $\chi(G) = \omega(G)$, then $\widehat{\alpha}(I) = \frac{\chi(G)}{\chi(G)-1}$.*

(ii) *If G is k-partite, then $\widehat{\alpha}(I) \geqslant \frac{k}{k-1}$.*

(iii) *If G is a complete k-partite graph, then $\widehat{\alpha}(I) = \frac{k}{k-1}$.*

(iv) *If G is bipartite, then $\widehat{\alpha}(I) = 2$.*

(v) *If $G = C_{2n+1}$, then $\widehat{\alpha}(I) = \frac{2n+1}{n+1}$.*

Thank you!